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THERMODYNAMICS AND THE GRIFFITH CRITERION FOR BRITTLE FRACTURE

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Abstract—The Griffith criterion for fracture, considered within the framework of nonlinear thermoelasticity, is shown to be a necessary condition for crack initiation, provided the underlying strain energy is properly interpreted, and provided the initial temperature is continuous at the crack tip.

NOTATION

Light-face letters indicate scalars; bold-face lower case letters indicate vectors (in $\mathscr{R}^2$); bold-face upper case letters indicate tensors (linear transformations of $\mathscr{R}^2$ into $\mathscr{R}^2$). $\mathbf{A}^T$ is the transpose of $\mathbf{A}$ and $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$. Here we have used standard indicial notation and Cartesian coordinates. Finally, div $\mathbf{S}$ is the vector field with components $\partial S_{ij}/\partial x_j$, while $\nabla\mathbf{u}$ is the tensor field with components $\partial u_i/\partial x_j$.

1. INTRODUCTION

The Griffith[1] criterion for fracture asserts that a crack propagates when the rate of decrease of strain energy with increased crack length is balanced by the concomitant increase in energy due to the formation of new crack surface.† Since this assertion is energetic in nature, it seems reasonable to expect at least a partial justification of the criterion within the framework of thermodynamics.‡ Such a partial justification is the main result of the present paper. In particular, we show that the Griffith criterion is, in a certain sense, *necessary* for crack initiation in an elastic body. Although we allow the temperature to vary with position and time, interestingly, the underlying strain energy turns out to be the free energy computed holding the temperature at its initial value.

Crucial to our analysis is the requirement that the temperature be continuous at the crack tip. Thus the above result will generally be inapplicable to situations in which, e.g. local plastic flow at the tip manifests itself, there, in a point source of heat and corresponding unbounded temperatures (See Ref. [6]).

To focus attention on the thermodynamics of crack propagation, we attempt to avoid geometrical and notational complications by limiting our discussion to edge cracks in two-dimensional bodies. The extension to more general types of cracks in two or three dimensions is not particularly difficult. Finally, while we do not rule out finite deformations, we do confine our attention to quasi-static behavior.

2. THE FIRST TWO LAWS FOR A REGULAR BODY

To fix notation we consider first a two-dimensional regular body $\mathscr{B}$, which we identify with the regular region of $\mathscr{R}^2$ it occupies in a fixed reference configuration. The first two laws for $\mathscr{B}$ take the form§

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}} \epsilon \,\mathrm{d}a &= -\int_{\partial\mathscr{B}} \mathbf{q}\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial\mathscr{B}} \mathbf{S}\boldsymbol{\nu}\cdot\dot{\mathbf{u}}\,\mathrm{d}s,\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}} \eta \,\mathrm{d}a &= -\int_{\partial\mathscr{B}} \frac{\mathbf{q}\cdot\boldsymbol{\nu}}{\theta}\,\mathrm{d}s + \int_{\mathscr{B}} \gamma \,\mathrm{d}a,\end{aligned} \tag{2.1}$$

where $\boldsymbol{\nu}$ is the outward unit normal to the boundary $\partial\mathscr{B}$ of $\mathscr{B}$. Here

†At least under fixed grip conditions in which the power expended on the body vanishes.

‡Other investigations involving the thermodynamics of fracture are Refs. [2–5].

§Note that we have restricted our attention to quasi-static processes, since we have neglected the rate of change of kinetic energy in $(2.1)_1$. Further, in writing (2.1) we have tacitly assumed that body forces and radiant heating are absent.

$\mathbf{u}(\mathbf{x}, t)$ is the displacement of the material point $\mathbf{x}$ (in $\mathscr{B}$) at time t;
$\mathbf{S}(\mathbf{x}, t)$ is the (Piola–Kirchhoff) stress;
$\epsilon(\mathbf{x}, t)$ is the internal energy;
$\eta(\mathbf{x}, t)$ is the entropy;
$\mathbf{q}(\mathbf{x}, t)$ is the heat flux;
$\theta(\mathbf{x}, t) > 0$ is the temperature;
$\gamma(\mathbf{x}, t) \geq 0$ is the entropy production.

Of course, (2.1) are valid not only for $\mathscr{B}$, but for any regular subregion of $\mathscr{B}$, and this fact, with the equation of equilibrium

$$\operatorname{div} \mathbf{S} = \mathbf{0}, \tag{2.2}$$

yields the balance equations

$$\begin{aligned} \dot{\epsilon} &= -\operatorname{div} \mathbf{q} + \mathbf{S} \cdot \nabla \dot{\mathbf{u}}, \\ \dot{\eta} &= -\operatorname{div} (\mathbf{q}/\theta) + \gamma. \end{aligned} \tag{2.3}$$

We restrict our attention to homogeneous† elastic materials; i.e. to materials defined by smooth‡ constitutive equations of the form

$$\begin{aligned} \mathbf{S} &= \hat{\mathbf{S}}(\mathbf{H}, \theta), \\ \epsilon &= \hat{\epsilon}(\mathbf{H}, \theta), \\ \eta &= \hat{\eta}(\mathbf{H}, \theta), \\ \mathbf{q} &= \hat{\mathbf{q}}(\mathbf{H}, \theta, \nabla\theta) \end{aligned} \tag{2.4}$$

with

$$\mathbf{H} = \nabla \mathbf{u}.$$

We define the free energy by

$$\psi = \hat{\psi}(\mathbf{H}, \theta) = \hat{\epsilon}(\mathbf{H}, \theta) - \theta\hat{\eta}(\mathbf{H}, \theta); \tag{2.5}$$

then, as is well known, compatibility with thermodynamics requires that§

$$\begin{aligned} \hat{\mathbf{S}} &= \frac{\partial \hat{\psi}}{\partial \mathbf{H}}, \qquad \hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta}, \\ \gamma &= -\frac{1}{\theta^2} \mathbf{q} \cdot \nabla\theta. \end{aligned} \tag{2.6}$$

3. PRELIMINARY DEFINITIONS

We assume that $\mathscr{B}$ contains an edge crack of negligible thickness. The crack is modeled as a smooth non-intersecting curve $\mathbf{z}(s)$ parametrized by arc length s, $0 \leq s \leq \ell(t)$; $s = 0$ corresponds to the intersection of the crack with the regular boundary¶ $\partial\mathscr{B}$ of $\mathscr{B}$; $s = \ell(t)$ labels the crack tip, which is allowed to advance smoothly with time. For convenience we write

$$\mathscr{C}(t) = \{\mathbf{z}(s) : 0 \leq s \leq \ell(t)\}$$

†All of our results are valid for inhomogeneous materials.
‡We use the term "smooth" as a synonym for "C^1".
§See Ref. [7]; here $\partial\hat{\psi}/\partial\mathbf{H}$ is the tensor with components $\partial\hat{\psi}/\partial H_{ij}$.
¶That is, the boundary $\mathscr{B}$ would have were it not cracked.

for the set of points comprising the crack,

$$\mathbf{z}_t = \mathbf{z}(\ell(t))$$

for the *position of the tip*, and

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_t$$

for the *velocity of propagation* of the tip. Since s measures arc length, $|\mathrm{d}\mathbf{z}/\mathrm{d}s| = 1$ and $|\mathbf{v}| = \dot{\ell}$.

We denote by $\mathbf{m}(s)$ a continuous unit vector field normal to the crack.

The fields $\varphi(\mathbf{x}, t)$ of interest will be defined at each $\mathbf{x}$ in $\mathscr{B}$-$\mathscr{C}(t)$ and each t in some time interval $[0, t_0)$. A field φ of this type is a C^n *fracture field* (see Ref. [8]), ($n \geq 0$ an integer) if:

(i) the derivatives of φ of order $\leq n$ exist away from the crack;

(ii) φ and its derivatives of order $\leq n$ are continuous away from the crack and, except at the tip, are continuous up to the crack from either side.

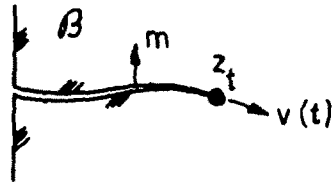

Fig. 1. Propagating crack.

Thus $\varphi(\mathbf{x}, t)$, as a function of $\mathbf{x}$, suffers (at most) a jump discontinuity across $\mathscr{C}(t)$–$\mathbf{z}_t$. We write

$$\varphi^{\pm}(s, t) = \lim_{\delta \to 0+} \varphi(\mathbf{z}(s) \pm \delta \mathbf{m}(s), t) \qquad (s \neq \ell(t))$$

for the values of φ on the upper and lower faces of the crack, and

$$[\varphi] = \varphi^{+} - \varphi^{-}$$

for the jump in φ across the crack.† In the above definition nothing is said about the behavior of φ at the crack tip (and, in fact, many of the thermodynamic fields are singular there).

Throughout this paper $\mathscr{D}_\delta(t)$ will denote a disc of radius δ centered at — and moving with — the crack tip, and

$$\mathscr{B}_\delta(t) = \mathscr{B} - \mathscr{D}_\delta(t).$$

Further, $\boldsymbol{\nu}$ will always denote the outward unit normal to $\mathscr{B}$, $\mathbf{n}$ the outward unit normal to $\partial\mathscr{D}_\delta$, and

$$v_n = \mathbf{v} \cdot \mathbf{n}$$

the normal velocity on $\partial\mathscr{D}_\delta$.

By a *fracture density* we mean a scalar-valued C^0 fracture field φ for which

$$\int_{\mathscr{B}} \varphi \, \mathrm{d}a \tag{3.1}$$

†Note that while, e.g. $[\mathbf{q}]$ depends on the choice of $\mathbf{m}$, $[\mathbf{q}] \cdot \mathbf{m}$ does not.

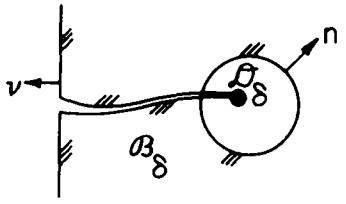

Fig. 2. Disc moving with tip.

exists as a Cauchy principal value; φ is *regular* if, in addition, φ is a C^1 fracture field, (3.1) is differentiable with respect to time, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}_\delta}\varphi\,\mathrm{d}a\to\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}\varphi\,\mathrm{d}a \tag{3.2}$$

as $\delta\to 0$.

By a *fracture flux* we mean a vector-valued C^0 fracture field $\mathbf{q}$ with the following properties:

(i) the integral

$$\int_0^{\ell(t)}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s \tag{3.3}$$

exists as a Cauchy principal value;

(ii) the limit as $\delta\to 0$ of

$$\int_{\partial\mathcal{D}_\delta}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\mathcal{S}$$

exists; it is called the *flow of* $\mathbf{q}$ *from the tip.*

The existence of (3.1) and (3.3) as Cauchy principal values simply means that

$$\begin{aligned}\int_{\mathcal{B}}\varphi\,\mathrm{d}a&=\lim_{\delta\to 0}\int_{\mathcal{B}_\delta}\varphi\,\mathrm{d}a,\\ \int_0^{\ell(t)}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s&=\lim_{\delta\to 0}\int_0^{s_\delta}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s,\end{aligned} \tag{3.4}$$

provided s_δ in $(0,\ell(t))$ tends to $\ell(t)$ as $\delta\to 0$.

Formally, let $\mathcal{S}(t)$ denote the *generalized boundary* of $\mathcal{B}$ at time t (i.e. the regular boundary $\partial\mathcal{B}$ plus the *two* faces of the crack), and let ν denote the outward unit normal to $\mathcal{S}(t)$. Clearly,

$$-\int_0^{\ell(t)}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s=\int_0^{\ell(t)}\mathbf{q}^+\cdot(-\mathbf{m})\,\mathrm{d}s+\int_0^{\ell(t)}\mathbf{q}^-\cdot\mathbf{m}\,\mathrm{d}s$$

represents the integral of $\mathbf{q}\cdot\nu$ over the two faces of the crack.

We therefore define

$$\int_{\mathcal{S}(t)}\mathbf{q}\cdot\nu\,\mathrm{d}\mathcal{S}=\int_{\partial\mathcal{B}}\mathbf{q}\cdot\nu\,\mathrm{d}\mathcal{S}-\int_0^{\ell(t)}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s. \tag{3.5}$$

The generalized boundary of $\mathcal{B}_\delta(t)$ consists of $\partial\mathcal{D}_\delta(t)$ plus a portion $\mathcal{S}_\delta(t)$ of the generalized boundary $\mathcal{S}(t)$ of $\mathcal{B}$. Guided by (3.5) we define

$$\int_{\mathcal{S}_\delta}\mathbf{q}\cdot\nu\,\mathrm{d}\mathcal{S}=\int_{\partial\mathcal{B}}\mathbf{q}\cdot\nu\,\mathrm{d}\mathcal{S}-\int_0^{s_\delta}[\mathbf{q}]\cdot\mathbf{m}\,\mathrm{d}s, \tag{3.6}$$

where $s = s_\delta(t)$ marks the intersection of the crack with $\partial\mathscr{D}_\delta(t)$. (Here we assume that δ is small enough that this intersection consists of a single point.) As a trivial consequence of $(3.4)_2$, (3.5) and (3.6) we have the following

Lemma. Let $\mathbf{q}$ *be a fracture flux. Then, as* $\delta \to 0$,

$$\int_{\mathscr{S}_\delta} \mathbf{q} \cdot \nu \, d\mathscr{H} \to \int_{\mathscr{S}} \mathbf{q} \cdot \nu \, d\mathscr{H}. \tag{3.7}$$

We are now in a position to state our assumptions concerning the thermodynamic fields $\mathbf{u}$, S, ϵ, η, $\mathbf{q}$, θ and γ.

(T_1) $\mathbf{u}$ and θ are C^2 fracture fields.

(T_2) ϵ and η are regular fracture densities.

(T_3) $\mathbf{S}^T\dot{\mathbf{u}}$, $\mathbf{q}$, and $\mathbf{q}/\theta$ are fracture fluxes.

(T_4) γ is a fracture density.

Of course, we also assume that the above fields satisfy the constitutive equations (2.4) and the field equations (2.2) and (2.3) away from the crack, and that the constitutive equations are consistent with the thermodynamic restrictions (2.6).

Hypotheses (T_2)–(T_4) limit the strength of the singularities at the crack tip. Intuition based on classical fracture mechanics tells us that the assumption regarding $\mathbf{S}^T\dot{\mathbf{u}}$ is reasonable, while the assumptions concerning ϵ and η are the weakest possible that lead to balance laws which make sense.

Proposition

$$\begin{aligned} \frac{d}{dt}\int_{\mathscr{B}_\delta} \epsilon \, da + \int_{\partial\mathscr{D}_\delta} \epsilon v_n \, d\mathscr{H} &= \int_{\mathscr{S}_\delta} (\mathbf{S}\nu \cdot \dot{\mathbf{u}} - \mathbf{q} \cdot \nu) \, d\mathscr{H} - \int_{\partial\mathscr{D}_\delta} (\mathbf{S}\mathbf{n} \cdot \dot{\mathbf{u}} - \mathbf{q} \cdot \mathbf{n}) \, d\mathscr{H}, \\ \frac{d}{dt}\int_{\mathscr{B}_\delta} \eta \, da + \int_{\partial\mathscr{D}_\delta} \eta v_n \, d\mathscr{H} &= -\int_{\mathscr{S}_\delta} \frac{\mathbf{q} \cdot \nu}{\theta} d\mathscr{H} + \int_{\partial\mathscr{D}_\delta} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} d\mathscr{H} + \int_{\mathscr{B}_\delta} \gamma \, da. \end{aligned} \tag{3.8}$$

Proof. By (2.2),

$$\operatorname{div}(\mathbf{S}^T\dot{\mathbf{u}}) = \mathbf{S} \cdot \nabla\dot{\mathbf{u}}. \tag{3.9}$$

Thus if we integrate (3.9) over $\mathscr{B}_\delta$ (whose "boundary" consists of $\partial\mathscr{B}$, $\partial\mathscr{D}_\delta$, and the two faces of the appropriate portion of the crack) and use the divergence theorem, we arrive at

$$\int_{\mathscr{B}_\delta} \mathbf{S} \cdot \nabla\dot{\mathbf{u}} \, da = \int_{\mathscr{S}_\delta} \mathbf{S}\nu \cdot \dot{\mathbf{u}} \, d\mathscr{H} - \int_{\partial\mathscr{D}_\delta} \mathbf{S}\mathbf{n} \cdot \dot{\mathbf{u}} \, d\mathscr{H}. \tag{3.10}$$

Similarly,

$$\int_{\mathscr{B}_\delta} \operatorname{div}\mathbf{q} \, da = \int_{\mathscr{S}_\delta} \mathbf{q} \cdot \nu \, d\mathscr{H} - \int_{\partial\mathscr{D}_\delta} \mathbf{q} \cdot \mathbf{n} \, d\mathscr{H},$$

and we conclude from $(2.3)_1$ that

$$\int_{\mathscr{B}_\delta} \dot{\epsilon} \, da$$

is equal to the right side of $(3.8)_1$. Equation $(3.8)_1$ therefore follows from the classical transport theorem (see Ref. [9], eqn (81.4))

$$\frac{d}{dt}\int_{\mathscr{B}_\delta} \epsilon \, da = \int_{\mathscr{B}_\delta} \dot{\epsilon} \, da - \int_{\partial\mathscr{D}_\delta} \epsilon v_n \, d\mathscr{H}. \tag{3.11}$$

The proof of $(3.8)_2$ is strictly analogous.

4. BEHAVIOR AT THE TIP

As the crack propagates energy and entropy are absorbed by the crack tip, and the first two laws for $\mathscr{B}$ must account for this fact. We therefore rewrite (2.1) in the forms

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}} \epsilon \,\mathrm{d}a + \mathscr{E} &= -\int_{\mathscr{S}} \mathbf{q}\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\mathscr{S}} \mathbf{S}\boldsymbol{\nu}\cdot\dot{\mathbf{u}}\,\mathrm{d}s, \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{B}} \eta \,\mathrm{d}a + \mathscr{N} &= -\int_{\mathscr{S}} \frac{\mathbf{q}\cdot\boldsymbol{\nu}}{\theta}\,\mathrm{d}s + \int_{\mathscr{B}} \gamma \,\mathrm{d}a, \end{aligned} \tag{4.1}$$

which may be interpreted as the defining relations for $\mathscr{E}(t)$ and $\mathscr{N}(t)$, the *rates at which energy and entropy are absorbed by the tip.*

When the only contributions to $\mathscr{E}$ and $\mathscr{N}$ are due to the formation of new surface, and when each new surface has constant surface energy and entropy, $\sigma/2$ and $\beta/2$, per unit length, then

$$\mathscr{E}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell} \sigma \,\mathrm{d}s = \sigma\dot{\ell},$$

$$\mathscr{N}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\ell} \beta \,\mathrm{d}s = \beta\dot{\ell}.$$

Here we will not find it necessary to assume that these relations hold.

Theorem. *As* $\delta \to 0$,

$$\begin{aligned} \int_{\partial\mathscr{B}_\delta} (\epsilon v_n + \mathbf{S}\mathbf{n}\cdot\dot{\mathbf{u}} - \mathbf{q}\cdot\mathbf{n})\,\mathrm{d}s &\to \mathscr{E}, \\ \int_{\partial\mathscr{B}_\delta} \left(\eta v_n - \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\right)\mathrm{d}s &\to \mathscr{N}. \end{aligned} \tag{4.2}$$

Proof. By $(3.8)_1$, (T_2), (3.2) (with $\varphi = \epsilon$), (T_3) and (3.7) (with $\mathbf{q}$ replaced by $\mathbf{q} - \mathbf{S}^{\mathrm{T}}\dot{\mathbf{u}}$),

$$\int_{\partial\mathscr{B}_\delta} \epsilon v_n \,\mathrm{d}s$$

must approach a limit as $\delta \to 0$, because all of the other terms in $(3.8)_1$ converge. Thus if we pass to this limit in $(3.8)_1$ and use $(4.1)_1$, we arrive at $(4.2)_1$. The proof of $(4.2)_2$ is similar; there we use $(3.8)_2$, $(4.1)_2$, and $(3.4)_1$ with $\varphi = \gamma$.

We now add two additional assumptions.

(T_5) $\theta(\mathbf{x}, 0)$ is continuous at the tip; i.e. there exists a number $\theta_T > 0$ such that $\theta(\mathbf{x}, 0) \to \theta_T$ whenever $\mathbf{x}$ tends to the tip from $\mathscr{B} - \mathscr{C}(0)$.

(T_6) At $t = 0$,

$$\int_{\partial\mathscr{B}_\delta} |\eta|\,\mathrm{d}s \quad \text{and} \quad \int_{\partial\mathscr{B}_\delta} |\mathbf{q}\cdot\mathbf{n}|\,\mathrm{d}s$$

remain bounded as $\delta \to 0$.

Assumption (T_5) is quite strong, since it most likely rules out the possibility of a heat source at the crack tip. Thus the present theory is most probably not a valid model for situations in which dissipative mechanisms, such as local plastic flow at the crack tip, are not negligible.
negligible.

The number

$$\Psi_0 = \mathscr{E}(0) - \theta_T \mathscr{N}(0) \tag{4.3}$$

represents the *initial rate at which free energy is absorbed by the tip.* Assumptions (T_5) and (T_6) allow us to establish a counterpart of (4.2) for Ψ_0.

Theorem. As $\delta \to 0$,

$$\left[\int_{\partial \mathscr{B}_\delta} (\psi v_n + \mathbf{S}\mathbf{n} \cdot \dot{\mathbf{u}})\, d\mathscr{A}\right]_{t=0} \to \Psi_0. \tag{4.4}$$

Proof. First of all, (T_5) implies that

$$\sup_{\mathbf{x} \in \mathscr{B}_\delta} |\theta(\mathbf{x}, 0) - \theta_T| \to 0$$

as $\delta \to 0$. Thus we may conclude from (T_6) that at $t = 0$,

$$\left| \int_{\partial \mathscr{B}_\delta} (\theta - \theta_T) \eta \mathbf{n}\, d\mathscr{A} \right| \leq \sup_{\mathscr{B}_\delta} |\theta - \theta_T| \int_{\partial \mathscr{B}_\delta} |\eta|\, d\mathscr{A} \to 0. \tag{4.5}$$

A similar argument shows that at $t = 0$,

$$\int_{\partial \mathscr{B}_\delta} \left(\frac{1}{\theta} - \frac{1}{\theta_T}\right) \mathbf{q} \cdot \mathbf{n}\, d\mathscr{A} \to 0. \tag{4.6}$$

If we multiply $(4.2)_2$ at $t = 0$ by θ_T and subtract the result from $(4.2)_1$ evaluated at the same time, and use (2.5), (4.3), (4.5) and (4.6), we arrive at (4.4).

Remark. It is important to note that—in contrast to (4.2)—the result (4.4) does not involve the heat flux **q**. Further, it is clear that if we were to require that (T_5) and (T_6) be satisfied for all t (with $\theta_T = \theta_T(t)$), then (4.4) (with $\Psi_0 = \Psi_0(t)$) would also hold for all t.

5. THE GRIFFITH CONDITION

We are now in a position to establish our main result—the necessity of the Griffith condition. To state this result concisely, we write

$$\theta_0(\mathbf{x}) = \theta(\mathbf{x}, 0)$$

for the initial temperature field, and

$$P_0 = \left[\int_{\mathscr{S}(t)} \mathbf{S}\boldsymbol{\nu} \cdot \dot{\mathbf{u}}\, d\mathscr{l}\right]_{t=0} \tag{5.1}$$

for the initial power expended on the body by the environment. We also use the abbreviation

$$\int_{\mathscr{B}} \hat{\psi}(\nabla \mathbf{u}(t), \theta_0)\, da = \int_{\mathscr{B}} \hat{\psi}(\nabla \mathbf{u}(\mathbf{x}, t), \theta_0(\mathbf{x}))\, da_{\mathbf{x}} \tag{5.2}$$

for the free energy the body would have at time t if the temperature were independent of time and equal to $\theta_0(\mathbf{x})$. Of course, (5.2) is not the actual free energy, since $\theta(\mathbf{x}, t)$, for $t > 0$, will generally be unequal to $\theta_0(\mathbf{x})$. To insure that (5.2) has meaning, we assume that

(T_7) $\hat{\psi}(\nabla \mathbf{u}, \theta_0)$ is a regular fracture density.

Theorem (Griffith criterion). *A necessary condition for crack propagation is that*

$$\left[\frac{d}{dt} \int_{\mathscr{B}} \hat{\psi}(\nabla \mathbf{u}(t), \theta_0)\, da\right]_{t=0} + \Psi_0 = P_0. \tag{5.3}$$

Proof. By (3.7) (with **q** replaced by $\mathbf{S}^T\dot{\mathbf{u}}$) and (3.10),

$$\lim_{\delta \to 0} \int_{\mathscr{B}_\delta} \mathbf{S} \cdot \nabla \dot{\mathbf{u}}\, da = \int_{\mathscr{S}} \mathbf{S}\boldsymbol{\nu} \cdot \dot{\mathbf{u}}\, d\mathscr{l} - \lim_{\delta \to 0} \int_{\partial \mathscr{B}_\delta} \mathbf{S}\mathbf{n} \cdot \dot{\mathbf{u}}\, d\mathscr{A}. \tag{5.4}$$

Next, by $(2.6)_1$,

$$(\hat{\psi}(\nabla \mathbf{u}, \theta_0))^{\cdot} = \hat{\mathbf{S}}(\nabla \mathbf{u}, \theta_0) \cdot \nabla \dot{\mathbf{u}}.$$

Thus if we apply the transport theorem (3.11) with ϵ replaced by $\hat{\psi}(\nabla \mathbf{u}, \theta_0)$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{R}_\delta} \hat{\psi}(\nabla \mathbf{u}, \theta_0)\, \mathrm{d}a = \int_{\mathscr{R}_\delta} \hat{\mathbf{S}}(\nabla \mathbf{u}, \theta_0) \cdot \nabla \dot{\mathbf{u}}\, \mathrm{d}a - \int_{\partial \mathscr{R}_\delta} \hat{\psi}(\nabla \mathbf{u}, \theta_0) v_n\, \mathrm{d}\mathbf{l}. \tag{5.5}$$

Note that at $t = 0$, $\hat{\mathbf{S}}(\nabla \mathbf{u}, \theta_0)$ is the initial stress and $\hat{\psi}(\nabla \mathbf{u}, \theta_0)$ is the initial free energy. Thus (5.1), (5.4) and (5.5) imply that†

$$\lim_{\delta \to 0} \left[\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{R}_\delta} \hat{\psi}(\nabla \mathbf{u}, \theta_0)\, \mathrm{d}a \right]_{t=0} = P_0 - \lim_{\delta \to 0} \left[\int_{\partial \mathscr{R}_\delta} (\psi v_n + \mathbf{S}\mathbf{n} \cdot \dot{\mathbf{u}})\, \mathrm{d}\mathbf{l} \right]_{t=0}.$$

But by (T_7) and (3.2) (with $\varphi = \hat{\psi}(\nabla \mathbf{u}, \theta_0)$),

$$\lim_{\delta \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{R}_\delta} \hat{\psi}(\nabla \mathbf{u}, \theta_0)\, \mathrm{d}a = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{R}} \hat{\psi}(\nabla u, \theta_0)\, \mathrm{d}a,$$

and we therefore conclude, with the aid of (4.4), that (5.3) holds. This completes the proof.

The last theorem is interesting: it asserts that the Griffith condition necessarily holds, at least for crack initiation, provided the underlying strain energy is the free energy of the body with the temperature held fixed at $\theta_0(\mathbf{x}) = \theta(\mathbf{x}, 0)$.

Remark. It is important to note when one is modeling situations in which entropy is produced at the tip, $\mathscr{N}$ will not represent a rate of increase of tip entropy. Indeed, in this instance $\mathscr{N} = \mathscr{N}_a - \Gamma$, where $\mathscr{N}_a(t)$ is the rate of increase of tip entropy and $\Gamma(t) \geq 0$ is the entropy production at the tip. Here it is more natural to define $\Psi_0 = \mathscr{E}(0) - \theta_T \mathscr{N}_a(0)$, in which case Ψ_0 in (5.3) should be replaced by $\Psi_0 + \theta_T \Gamma(0)$. Of course, the assumption (T_5) would probably preclude a production of entropy at the tip.

†Here the limits *follow* evaluation at $t = 0$. The limit involving ψ on the right side exists because all of the other limits exist.

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